EUROPHYSICS LETTERS 10 February 1997

Europhys. Lett., **37** (5), pp. 347-352 (1997)

Magnetization of mesoscopic disordered networks

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(received 29 November 1996; accepted 10 January 1997)

PACS. 72.10Bg – General formulation of transport theory.
PACS. 05.30Fk – Fermion systems and electron gas.
PACS. 71.23Cq – Amorphous semiconductors, metallic glasses, glasses.

Abstract. – We study the magnetic response of mesoscopic metallic isolated networks. We calculate the average and typical magnetizations in the diffusive regime for non-interacting electrons or in the first-order Hartree-Fock approximation. These quantities are related to the return probability for a diffusive particle on the corresponding network. By solving the diffusion equation on various types of networks, including a ring with arms, an infinite square network or a chain of connected rings, we deduce the corresponding magnetizations. In the case of an infinite network, the Hartree-Fock average magnetization stays finite in the thermodynamic limit. We discuss the relevance of our results to the experimental situation. Quite generally, when rings are connected, the average magnetization is only weakly reduced by a numerical factor.

The problem of persistent currents in mesoscopic rings [1] has been stimulated by a few key experiments in the recent years. Two types of measurements have been made, single-ring experiments [2]-[4] and many-ring experiments [4]-[6]. In the second case, the measured quantity is an average magnetization $\langle M \rangle$, while the first type of experiment can only give the magnetization corresponding to a given disorder configuration. In the last case, the width M_{typ} of the magnetization distribution is also of interest: $M_{typ}^2 = \langle M^2 \rangle - \langle M \rangle^2$.

To describe these currents, two types of methods have been used: i) analytical methods where the diffusive electronic motion is treated in a perturbative way, leading to the Cooperon diagrams; non-interacting electron theory [7]-[11] or Hartree-Fock approximation [9], [10], [12]-[14] have been considered; ii) strictly 1D models or numerical methods in which either there is no diffusive motion or the system size is too small to give quantitative results [15]. Up to now, the only results for the diffusive regime are given by the perturbative method. The status of the comparison with experiments is not completely clear yet [16]. Most recent data [4] are in reasonably good agreement with theory, the typical current being described by the non-interacting theory [7], [11], [16] and the average current by the Hartree-Fock approximation [12]-[14], [16].

We propose that a new way to get insight into the problem is to study *other geometries* than simple rings. Such a situation has actually been already realized in the experiment of ref. [3]

where the ring is connected to two arms. In this letter, we calculate analytically the typical and average magnetizations of various types of networks, following the method i). To do so, we use a semi-classical picture to relate the quantities of interest to the return probability of a classically diffusive particle. Then, this return probability is calculated on these different networks, giving access to the magnetization. As examples, we treat the cases of an isolated ring connected to one or two arms, of an infinite square lattice and of a chain of rings. Several experiments are proposed.

In the absence of e-e interactions, a finite contribution to the average magnetization comes from the fact that the number N of particles is fixed in each subsystem of the ensemble [17]. It turns out that this contribution is by far smaller than the experimental results. However, we will discuss it mainly for pedagogical purpose and comparison with other contributions. With this constraint on N, the "canonical" magnetization is given by [18]

$$\langle M_N(H)\rangle = -\frac{\Delta}{2}\frac{\partial}{\partial H}\langle \delta N^2(\mu)\rangle\,, \tag{1}$$

where Δ is the mean level spacing and $\langle \delta N^2(\mu)\rangle$ is the sample-to-sample fluctuation of the number of single-particle states below the Fermi energy μ. It is an integral of the two-point correlation function of the density of states (DOS) $K(\varepsilon_1-\varepsilon_2) = \langle \rho(\varepsilon_1)\rho(\varepsilon_2)\rangle - \rho_0^2$. ρ_0 is the average DOS. $K(\varepsilon)$ has been calculated by Altshuler and Shklovskii [19] and later in the presence of a magnetic flux [9], [8]. A very useful semi-classical picture has been presented by Argaman *et al.*, which relates the Fourier transform $\tilde{K}(t)$ of $K(\varepsilon)$ to the classical return probability $p(\mathbf{r},\mathbf{r},t)$ for a diffusive particle [11]: $\tilde{K}(t) = tP(t)/(4\pi^2)$, where $P(t) = \int p(\mathbf{r},\mathbf{r},t)\mathrm{d}\mathbf{r}$ (otherwise specified, $\hbar = 1, c = 1$ throughout the paper). This return probability has two components, the purely classical one and the interference term which results from interferences between pairs of time-reversed trajectories. In the diagrammatic picture, they are related to the diffuson and Cooperon diagrams. The interference term is field dependent and the Fourier transform $p_\gamma(\mathbf{r},\mathbf{r}',\omega)$ is solution of the diffusion equation

$$[\gamma - i\omega - D(\nabla + 2ie\mathbf{A})^2]p_\gamma(\mathbf{r},\mathbf{r}',\omega) = \delta(\mathbf{r}-\mathbf{r}')\,, \tag{2}$$

where $\gamma = D/L_\phi^2$ is the inelastic-scattering rate. L_ϕ is the phase coherence length.

From eqs. (1) and from the above expression of the form factor, the average canonical magnetization can be related to the field-dependent part of the return probability:

$$\langle M_N(H)\rangle = -\frac{\Delta}{4\pi^2}\frac{\partial}{\partial H}\int_0^\infty \frac{P_\gamma(t,H)}{t}\mathrm{d}t\,. \tag{3}$$

Note that the field-dependent part of this integral converges at small times. At large times, the return probability is exponentially cut off as $e^{-\gamma t}$.

Due to the e-e interactions, a larger contribution to the average magnetization exists, which has been calculated in the Hartree-Fock approximation [12]. It can be written as [9], [12]-[14]

$$\langle M_{\mathrm{ee}}(H)\rangle = -\frac{U}{4}\frac{\partial}{\partial H}\int \langle n(\mathbf{r})^2\rangle \mathrm{d}\mathbf{r}\,, \tag{4}$$

where U is an effective screened interaction and $n(\mathbf{r})$ is the local density for the non-interacting system. The integrand is related to the fluctuations of the local DOS which in turn can be related to the return probability [14]. One gets

$$\langle M_{\mathrm{ee}}(H)\rangle = -\frac{U\rho_0}{\pi}\frac{\partial}{\partial H}\int_0^\infty \frac{P_\gamma(t,H)}{t^2}\mathrm{d}t\,. \tag{5}$$

In a similar way, the typical magnetization can also be straightforwardly written in terms of $K(\varepsilon)$ [11], [16]. By Fourier transform, one has

$$M_{typ}^2(H) = \frac{1}{8\pi^2}\int_0^\infty \frac{P''_\gamma(t,H)|_0^H}{t^3}\mathrm{d}t\,, \tag{6}$$

where $P''_\gamma(t,H)|_0^H = \partial^2 P_\gamma/\partial H^2|_0 - \partial^2 P_\gamma/\partial H^2|_H$.

To be complete, we recall that the weak-localization correction to the conductance of a connected mesoscopic sample can also be related to the return probability [20], [21]: $\Delta\sigma(r) = (-2/\pi\rho_0)\sigma_0 C_\gamma(\mathbf{r},\mathbf{r})$, where σ_0 is the Drude conductivity. The Cooperon $C_\gamma(\mathbf{r},\mathbf{r},H)$ is the time-integrated field-dependent return probability: $C_\gamma(\mathbf{r},\mathbf{r},H) = \int_0^\infty p_\gamma(r,r,t,H)\mathrm{d}t$. It appears that all the quantities of interest are obtained as time integrals of the return probability with various power law weighting functions. Noting that $P_\gamma(t)$ has the form $P_0(t)e^{-\gamma t}$ and that

$$\int \frac{P_0(t)}{t}e^{-\gamma t}\mathrm{d}t = \int_\gamma^\infty \mathrm{d}\gamma \int C_\gamma(\mathbf{r},\mathbf{r},H)\mathrm{d}\mathbf{r}\,, \tag{7}$$

the different magnetizations can be given in terms of the successive integrals of $C_\gamma(\mathbf{r},\mathbf{r},H)$:

$$\langle M_N(H)\rangle = -\frac{\Delta}{4\pi^2}\frac{\partial}{\partial H}\int C_\gamma^{(1)}(\mathbf{r},\mathbf{r},H)\mathrm{d}\mathbf{r}\,, \tag{8}$$

$$\langle M_{\mathrm{ee}}(H)\rangle = -\frac{U\rho_0}{\pi}\frac{\partial}{\partial H}\int C_\gamma^{(2)}(\mathbf{r},\mathbf{r},H)\mathrm{d}\mathbf{r}\,, \tag{9}$$

$$M_{typ}^2(H) = \frac{1}{8\pi^2}\frac{\partial^2}{\partial H^2}\int C_\gamma^{(3)}(\mathbf{r},\mathbf{r},H)\big|_0^H\mathrm{d}\mathbf{r}\,, \tag{10}$$

where $C_\gamma^{(n)} = \int_\gamma^\infty \mathrm{d}\gamma_n \ldots \int_{\gamma_2}^\infty \mathrm{d}\gamma_1 \int_{\gamma_1}^\infty \mathrm{d}\gamma' C_{\gamma'}$. These are the key equations of this paper since the different magnetizations can be calculated from the knowledge of the return probability $C_\gamma(\mathbf{r},\mathbf{r},H)$ on the different lattices considered and can be deduced from each other or related to weak-localization correction by H- or γ-derivatives or integrations.

For the case of weak-localization correction, an extensive study of this quantity on various lattices has been carried out by Douçot and Rammal [21]. Considering networks made of quasi-1D wires, so that the diffusion can be considered as one-dimensional, the Cooperon $C_\gamma(r,r')$ obeys the one-dimensional diffusion equation

$$[\gamma - D(\nabla + 2ieA)^2]C_\gamma(r,r') = \delta(r-r') \tag{11}$$

with the continuity equations written for every node α (including the starting point r' that can be considered as an additional node in the lattice) [21]

$$\sum_\beta \Big(-i\frac{\partial}{\partial r} - \frac{2eA}{\hbar c}\Big)C_\gamma(r,r')|_{r=\alpha} = \frac{i}{DS}\delta_{r',\alpha}\,; \tag{12}$$

r, r' are linear coordinates on the network. The sum is taken over all links relating the node α to its neighboring nodes β. Integration of the differential equation (11) with the boundary conditions (12) leads to the so-called network equations which relate $C_\gamma(\alpha,r')$ to the neighboring $C_\gamma(\beta,r')$:

$$\sum_\beta \coth\Big(\frac{l_{\alpha\beta}}{L_\phi}\Big)C(\alpha,r') - \sum_\beta \frac{C(\beta,r')e^{-i\gamma_{\alpha\beta}}}{\sinh(l_{\alpha\beta}/L_\phi)} = \frac{L_\phi}{DS}\delta_{\alpha,r'}\,; \tag{13}$$

$l_{\alpha\beta}$ is the length of the link $(\alpha\beta)$ and $\gamma_{\alpha\beta} = (4\pi/\phi_0)\int_\alpha^\beta \mathbf{A}\mathrm{d}\mathbf{l}$ is the circulation of the vector potential along this link. Finally, spatial integration of $C_\gamma(r', r')$ gives access to the magnetizations.

As an example, we have first considered the case of a ring of perimeter L connected to an arm of length b. Such a geometry has been considered in the strictly 1D case without disorder [22]-[24]. It is expected that since the electrons will spend some time in the arm where they are not sensitive to the flux, the persistent current will be decreased. From eqs. (11), (13), the function $C_\gamma(r, r)$ can be straightforwardly calculated on the arm and on the ring. Spatial integration gives

$$\langle M_N \rangle = \frac{\Delta S}{\pi\phi_0} \frac{\sin 4\pi\varphi}{\frac{1}{2}\tanh\frac{b}{L_\phi}\sinh\frac{L}{L_\phi} + \cosh\frac{L}{L_\phi} - \cos 4\pi\varphi}\,,$$

where $\varphi = \Phi/\phi_0$. Φ is the flux through the ring, $\phi_0 = h/e$ is the flux quantum and S is the area of the ring. $\langle M_{\mathrm{ee}} \rangle$ and M_{typ} are given by successive integrations over γ according to eqs. (8)-(10). In the limit $b \to \infty$, the interlevel spacing $\Delta(b) = \Delta(0)L/(L+b)$ tends to zero and the canonical magnetization $\langle M_N \rangle$ vanishes. More interestingly, it is seen that the magnetizations $\langle M_{\mathrm{ee}} \rangle$ and M_{typ} do not decrease to 0 when $b \ll L_\phi$ but they *saturate* to finite values with respective reductions of the m-th harmonics in the ratios $(2/3)^m$ and $(2/3)^{m/2}$.

The case of a ring connected to two diametrically opposite arms, as in the experiment of ref. [3], can be treated in a very similar way. In this case we find that for infinite arms ($b \gg L_\phi$), the lowest harmonics of the e-e average current are reduced in a ratio 4/9 and the typical current is reduced by a factor 2/3. This result is relevant for the experiment of ref. [3] where the magnetization is measured for open and connected rings. Moreover, in the limit $b \gg L_\phi$, the magnetization should be unchanged if reservoirs are attached to the arms [22]. We propose that single-ring experiments with appropriately designed arms could be able to measure these reductions.

We now turn to the case of an infinite square lattice whose magnetization will be compared with the one of an array of isolated rings. The eigenvalues of the diffusion equation can be calculated for a rational flux per plaquette $\varphi = Ha^2/\phi_0 = p/2q$. a is the lattice parameter. Defining $\eta = a/L_\phi$, we find that the canonical magnetization per plaquette is

$$\langle M_N \rangle = \frac{\Delta}{4\pi^2 q}\frac{\partial}{\partial H}\sum_{i=1}^{q}\langle\langle \ln(4\cosh\eta - \varepsilon_i(\theta,\mu))\rangle\rangle\,, \tag{14}$$

where $\langle\langle(\ldots)\rangle\rangle = \int_0^{2\pi}\frac{\mathrm{d}\theta}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\mu}{2\pi}(\ldots)$. $\varepsilon_i(\theta,\mu)$ are the solutions of the determinantal equation $\det M = 0$, where the $q \times q$ matrix M is defined by $M_{nn} = 2\cos(4n\pi\varphi + \theta/q) - \varepsilon$, $M_{n,n+1} = M_{n+1,n} = 1$ for $n \le q-1$ and $M_{1,q} = M^*_{q,1} = \exp[i\mu]$. This is the matrix associated to the Harper equation known to be also relevant for other related problems like tight-binding electrons in a magnetic field [25] or superconducting networks in a field [26].

The magnetization per plaquette can be compared to the magnetization of a square ring of perimeter $L = 4a$:

$$\langle M_N \rangle = \frac{\Delta}{4\pi^2}\frac{\partial}{\partial H}\ln(\cosh 4\eta - \cos 4\pi\varphi)\,. \tag{15}$$

Since $\Delta \to 0$ for the infinite network, this canonical magnetization density vanishes for an infinite network, as was already noticed for a chain of connected rings [27]. On the other hand, the e-e contribution stays finite in the thermodynamic limit. It is given by

$$\langle M_{\mathrm{ee}} \rangle = U\rho_0\frac{eD}{\pi^2 q}\frac{\partial}{\partial\varphi}\sum_{i=1}^{q}\int_\eta^\infty\{\ln(4\cosh\eta - \varepsilon_i(\theta,\mu))\}\eta\mathrm{d}\eta$$

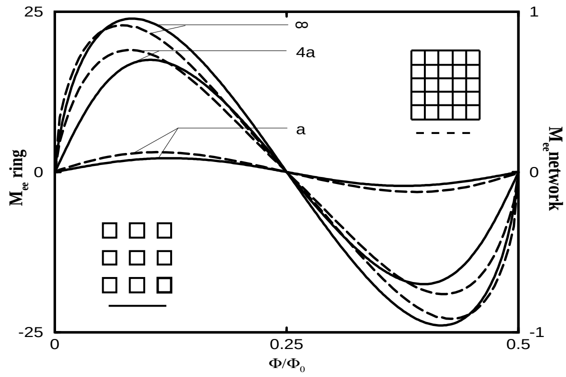

Fig. 1. – Average magnetization $\langle M_{\mathrm{ee}}\rangle$ of a single ring (full lines) and magnetization density of the infinite network (dashed lines), for $L_\phi = \infty$, $4a$ and a.

and can be compared with the ring magnetization which can be cast in the form

$$\langle M_{\mathrm{ee}}\rangle = U\rho_0 \frac{4eD}{\pi}\int_\eta^\infty \frac{\sin 4\pi\varphi}{\cosh 4\eta - cos 4\pi\varphi}\eta\, \mathrm{d}\eta\,. \tag{16}$$

(This integral can be calculated explicitly in terms of the Lobatchevsky function and it has the Fourier decomposition found by Ambegaokar and Eckern [12].) Contrary to the canonical magnetization, the e-e magnetization is an extensive quantity. This magnetization density is plotted in fig. 1 for the ring and the infinite lattice. It is first seen that the network magnetization is continuous. Although the field dependence of the eigenvalues of the Harper equation has a very complicated discontinuous behavior (the so-called Hofstadter spectrum), the sum on the eigenvalues has a smooth behavior [28]. $\langle M_{\mathrm{ee}}\rangle$ can be easily calculated for large q. In this case the dispersion $\varepsilon_i(\theta,\mu)$ is very small and the density of states can be approximated by a sum of δ functions [29].

It is seen in fig. 1 that the network magnetization density is about 25 times smaller than the ring magnetization. Considering that, on the array of square rings already considered experimentally [5], the distance between rings is equal to the size of the squares, the number of squares is four times larger when they are connected. One then expects only a factor of order 6 between the magnetization of the array of disconnected rings and the lattice. The width of the magnetization distribution scales as $1/\sqrt{S}$, S being the area of the network.

We have also calculated the magnetization of a chain of rings connected with arms of similar length, an obvious generalization of the experiment done in ref. [4], and find that when the rings are connected the average Hartree-Fock magnetization is reduced by a factor 3. As before, the canonical magnetization vanishes. The result of this experiment would immediately tell about the importance of the interactions.

Our results have been obtained in the Hartree-Fock approximation and should be corrected by higher-order contributions [30]. However, the ratio between magnetizations of connected and disconnected rings should stay unchanged. In conclusion, we have calculated the magnetization of various mesoscopic networks. Comparison with experiments should probe the importance of e-e interactions and their interplay with the diffusive nature of the electronic motion.

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Part of the work was done at the Institute of Theoretical Physics, UCSB, and supported by the NSF Grant No. PHY94-07194.

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